1 The Problem

You're on a two-dimensional grid $\mathbb{Z} \times \mathbb{Z}$ and have to find a way to get to one coordinate $(x, y) \in \mathbb{Z} \times \mathbb{Z}$. You start at $(0, 0)$.

In your i-th step you move either $\underbrace{(+i, 0)}_{=:E}$, $\underbrace{(-i, 0)}_{=:W}$, $\underbrace{(0, +i)}_{=:N}$ or $\underbrace{(0, -i)}_{=:S}$.

2 The algorithm

Algorithm 1 Algorithm to calculate the minimum amount of steps

function CALCULATESTEPS($x \in \mathbb{Z}$, $y \in \mathbb{Z}$)

  $s \leftarrow 1$

  $dist \leftarrow |x| + |y|$

  while $\overbrace{\frac{s^2+s}{2} < dist}^{\text{condition 1}}$ or $\overbrace{\frac{s^2+s}{2} \not\equiv dist \pmod 2}^{\text{condition 2}}$ **do**

    $s \leftarrow s + 1$

  end while

  return s

end function

Algorithm 2 Algorithm to solve the pogo problem

```
function solvePogo(x ∈ ℤ, y ∈ ℤ)
    max ← calculateSteps(x, y)

    solution ← ε
    for i in max, ..., 1 do
        if |x| > |y| then
            if x > 0 then
                solution ← solution + E
                x ← x − i
            else
                solution ← solution + W
                x ← x + i
            end if
        else
            if y > 0 then
                solution ← solution + N
                y ← y − i
            else
                solution ← solution + S
                y ← y + i
            end if
        end if
    end for

    return solution
end function
```

3 Correctness

3.1 calculateSteps

Let $x, y \in \mathbb{Z}$ and $s := \textsc{calculateSteps}(x, y)$.

Let s_{min} be the minimum amount of necessary steps to get from $(0,0)$ to (x,y) when you move i units in your i'th step.

Theorem: $s = s_{\text{min}}$

It's enough to proof $s \geq s_{\text{min}}$ and $s \leq s_{\text{min}}$.

Theorem: $s \leq s_{\text{min}}$ (we don't make too many steps)

Proof:

We have to get from $(0,0)$ to (x,y). As we may only move in taxicab geometry we have to use the taxicab distance measure d_1:

$$d_1(p, q) := \sum_{i=1}^{2} |p_i - q_i|$$

So in our scenario:

$$d_1((0,0),(x,y)) = |x| + |y|$$

This means we have to move at least $|x| + |y|$ units to get from $(0,0)$ to (x,y). As we move i units in the i'th step, we have to solve the following equations for $s_{\min 1}$:

$$\sum_{i=1}^{s_{\min 1}} i \geq |x| + |y| \qquad \text{and} \qquad |x| + |y| > \sum_{i=1}^{s_{\min 1}-1} i \tag{1}$$

$$\frac{s_{\min 1}^2 + s_{\min 1}}{2} \geq |x| + |y| \qquad\qquad > \sum_{i=1}^{s_{\min 1}-1} i \tag{2}$$

This is what algorithm 1 check with condition 1. As the algorithm increases s only by one in each loop, it makes sure that $\sum_{i=1}^{s_{\min 1}-1} i$ is bigger than $|x| + |y|$.

TODO: Proof necessarity of condition two

TODO: I guess I should initialize s with 0 (should only make a difference when (x,y) = (0,0))

Theorem: $s \geq s_{\min}$ (we make enough steps)

Proof:

TODO

3.2 solvePogo

Theorem: SOLVEPOGO(x, y) returns a valid, minimal sequence of steps to get from $(0,0)$ to (x,y)

Proof:

TODO