Minimal distance to a cubic function

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When you want to develop a selfdriving car, you have to plan which path it should take. A reasonable choice for the representation of paths are cubic splines. You also have to be able to calculate how to steer to get or to remain on a path. A way to do this is applying the PID algorithm. This algorithm needs to know the signed current error. So you need to be able to get the minimal distance of a point to a cubic spline combined with the direction (left or right). As you need to get the signed error (and one steering direction might be prefered), it is not only necessary to get the minimal absolute distance, but also to get all points on the spline with minimal distance.

In this paper I want to discuss how to find all points on a cubic function with minimal distance to a given point. As other representations of paths might be easier to understand and to implement, I will also cover the problem of finding the minimal distance of a point to a polynomial of degree 0, 1 and 2.

1 Description of the Problem

Let $f : \mathbb{R} \to \mathbb{R}$ be a polynomial function and $P \in \mathbb{R}^2$ be a point. Let $d_{P,f} : \mathbb{R} \to \mathbb{R}_0^+$ be the Euklidean distance $d_{P,f}$ of a point P and a point $(x, f(x))$:

$$d_{P,f}(x) := \sqrt{(x_P - x)^2 + (y_P - f(x))^2}$$

Now there is finite set $x_1, \dots, x_n$ such that

Should I proof this?

$$\forall \tilde{x} \in \mathbb{R} \setminus \{x_1, \dots, x_n\} : d_{P,f}(x_1) = \dots = d_{P,f}(x_n) < d_{P,f}(\tilde{x})$$

Essentially, you want to find the minima $x_1, \dots, x_n$ for given f and P. But minimizing $d_{P,f}$ is the same as minimizing $d_{P,f}^2$:

$$\begin{aligned} d_{P,f}(x)^2 &= \sqrt{(x_P - x)^2 + (y_P - f(x))^2}^2 && (1) \\ &= x_p^2 - 2x_p x + x^2 + y_p^2 - 2y_p f(x) + f(x)^2 && (2) \end{aligned}$$

Hat dieser Satz einen Namen? Gibt es ein gutes Buch, aus dem ich den zitieren kann? Ich habe ihn aus meinem Analysis I Skript (Satz 21.5).

Theorem 1
Let x_0 be a relative extremum of a differentiable function $f : \mathbb{R} \to \mathbb{R}$.

Then: $f'(x_0) = 0$.

2 Minimal distance to a constant function

Let $f(x) = c$ with $c \in \mathbb{R}$ be a constant function.

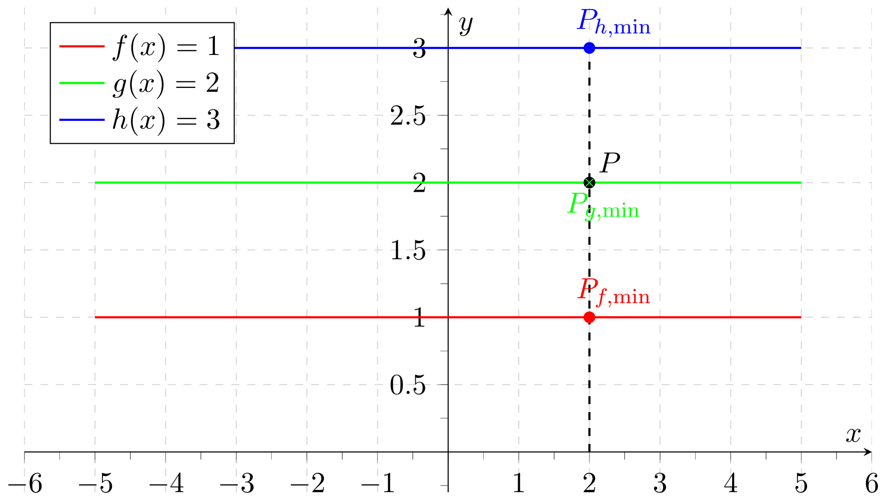

Figure 1: Three constant functions and their points with minimal distance

Then $(x_P, f(x_P))$ has minimal distance to P. Every other point has higher distance. See Figure 1.

3 Minimal distance to a linear function

Let $f(x) = m \cdot x + t$ with $m \in \mathbb{R} \setminus \{ 0 \}$ and $t \in \mathbb{R}$ be a linear function.

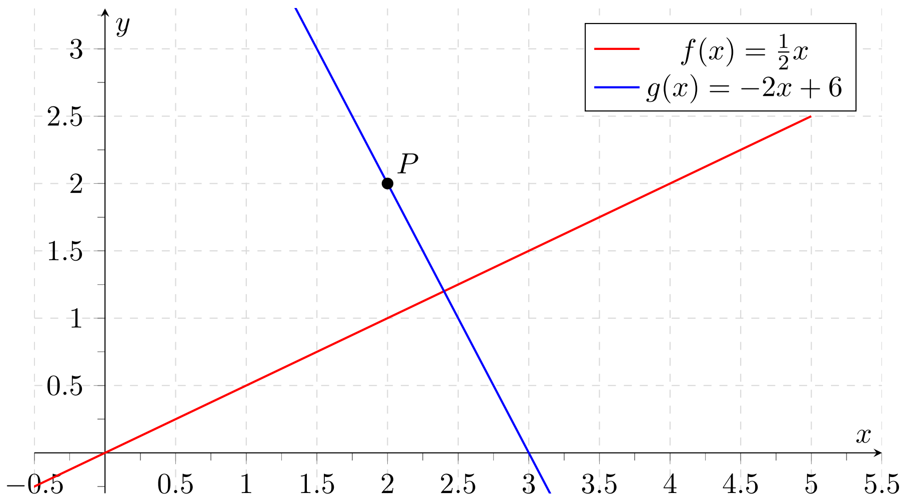

Figure 2: The shortest distance of P to f can be calculated by using the perpendicular

Now you can drop a perpendicular $f_\perp$ through P on $f(x)$. The slope of $f_\perp$ is $-\frac{1}{m}$. Now you can calculate $f_\perp$:

$$f_\perp(x) = -\frac{1}{m} \cdot x + t_\perp \tag{3}$$

$$\Rightarrow y_P = -\frac{1}{m} \cdot x_P + t_\perp \tag{4}$$

$$\Leftrightarrow t_\perp = y_P + \frac{1}{m} \cdot x_P \tag{5}$$

Now find the point $(x, f(x))$ where the perpendicular crosses the function:

$$f(x) = f_\perp(x) \tag{6}$$

$$\Leftrightarrow m \cdot x + t = -\frac{1}{m} \cdot x + \left(y_P + \frac{1}{m} \cdot x_P\right) \tag{7}$$

$$\Leftrightarrow \left(m + \frac{1}{m}\right) \cdot x = y_P + \frac{1}{m} \cdot x_P - t \tag{8}$$

$$\Leftrightarrow x = \frac{m}{m^2+1}\left(y_P + \frac{1}{m} \cdot x_P - t\right) \tag{9}$$

There is only one point with minimal distance. See Figure 2.

4 Minimal distance to a quadratic function

Let $f(x) = a \cdot x^2 + b \cdot x + c$ with $a \in \mathbb{R} \setminus \{ 0 \}$ and $b, c \in \mathbb{R}$ be a quadratic function.

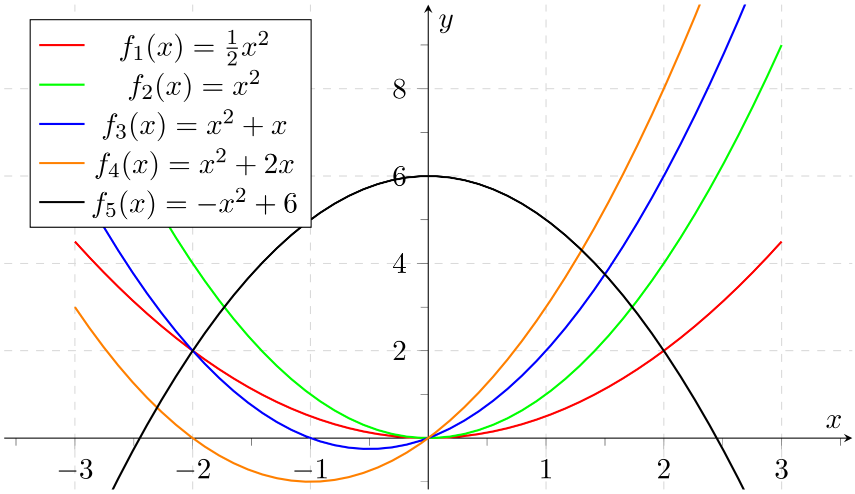

Figure 3: Quadratic functions

4.1 Calculate points with minimal distance

In this case, $d_{P,f}^2$ is polynomial of degree 4. We use Theorem 1:

$$\begin{aligned}
0 &\stackrel{!}{=} (d_{P,f}^2)' && (10)\\
&= -2x_p + 2x - 2y_p f'(x) + \big(f(x)^2\big)' && (11)\\
&= -2x_p + 2x - 2y_p f'(x) + 2f(x) \cdot f'(x) \qquad \text{(chain rule)} && (12)\\
\Leftrightarrow 0 &\stackrel{!}{=} -x_p + x - y_p f'(x) + f(x) \cdot f'(x) \qquad \text{(divide by 2)} && (13)\\
&= -x_p + x - y_p(2ax + b) + (ax^2 + bx + c)(2ax + b) && (14)\\
&= -x_p + x - y_p \cdot 2ax - y_p b + (2a^2x^3 + 2abx^2 + 2acx + abx^2 + b^2x + bc) && (15)\\
&= -x_p + x - 2y_p ax - y_p b + (2a^2x^3 + 3abx^2 + 2acx + b^2x + bc) && (16)\\
&= 2a^2x^3 + 3abx^2 + (1 - 2y_p a + 2ac + b^2)x + (bc - by_p - x_p) && (17)
\end{aligned}$$

This is an algebraic equation of degree 3. There can be up to 3 solutions in such an equation. Those solutions can be found with a closed formula.

Where are those closed formulas?

Example 1

Let $a = 1, b = 0, c = 1, x_p = 0, y_p = 1$. So $f(x) = x^2 + 1$ and $P(0, 1)$.

$$\begin{aligned}
0 &\stackrel{!}{=} 4x^3 - 2x && (18)\\
&= 2x(2x^2 - 1) && (19)\\
\Rightarrow x_1 = 0 \quad x_{2,3} &= \pm\frac{1}{\sqrt{2}} && (20)
\end{aligned}$$

As you can easily verify, only x_1 is a minimum of $d_{P,f}$.

4.2 Number of points with minimal distance

Theorem 2
A point P has either one or two points on the graph of a quadratic function f that are closest to P.

In the following, I will do some transformations with $f = f_0$ and $P = P_0$.

Moving f_0 and P_0 simultaneously in x or y direction does not change the minimum distance. Furthermore, we can find the points with minimum distance on the moved situation and calculate the minimum points in the original situation.

First of all, we move f_0 and P_0 by $\frac{b}{2a}$ in x direction, so

$$f_1(x) = ax^2 - \frac{b^2}{4a} + c \quad \text{and} \quad P_1 = \left(x_p + \frac{b}{2a},\ y_p\right)$$

Because:

$$\begin{aligned} f(x - b/2a) &= a(x - b/2a)^2 + b(x - b/2a) + c && (21)\\ &= a(x^2 - b/ax + b^2/4a^2) + bx - b^2/2a + c && (22)\\ &= ax^2 - bx + b^2/4a + bx - b^2/2a + c && (23)\\ &= ax^2 - b^2/4a + c && (24) \end{aligned}$$

Then move f_1 and P_1 by $\frac{b^2}{4a} - c$ in y direction. You get:

$$f_2(x) = ax^2 \quad \text{and} \quad P_2 = \left(x_p + \frac{b}{2a},\ y_p + \frac{b^2}{4a} - c\right)$$

Case 1: As $f_2(x) = ax^2$ is symmetric to the y axis, only points $P = (0, w)$ could possilby have three minima.

Then compute:

$$\begin{aligned} d_{P,f_2}(x) &= \sqrt{(x - x_P)^2 + (f(x) - w)^2} && (25)\\ &= \sqrt{x^2 + (ax^2 - w)^2} && (26)\\ &= \sqrt{x^2 + a^2x^4 - 2awx^2 + w^2} && (27)\\ &= \sqrt{a^2x^4 + (1 - 2aw)x^2 + w^2} && (28)\\ &= \sqrt{\left(a^2x^2 + \frac{1 - 2aw}{2}\right)^2 + w^2 - (1 - 2aw)^2} && (29)\\ &= \sqrt{(a^2x^2 + 1/2 - aw)^2 + (w^2 - (1 - 2aw)^2)} && (30)\\ & && (31) \end{aligned}$$

The term

$$a^2x^2 + (1/2 - aw)$$

should get as close to 0 as possilbe when we want to minimize d_{P,f_2}. For $w \leq 1/2a$ you only have $x = 0$ as a minimum. For all other points $P = (0, w)$, there are exactly two minima $x_{1,2} = \pm\sqrt{aw - 1/2}$.

Case 2: $P = (z, w)$ is not on the symmetry axis, so $z \neq 0$. Then you compute:

$$\begin{aligned}
d_{P,f_2}(x) &= \sqrt{(x-z)^2 + (f(x)-w)^2} && (32)\\
&= \sqrt{(x^2 - 2zx + z^2) + ((ax^2)^2 - 2awx^2 - w^2)} && (33)\\
&= \sqrt{a^2x^4 + (1-2aw)x^2 + (-2z)x + z^2 - w^2} && (34)\\
0 &\overset{!}{=} d'_{P,f_2}(x) && (35)\\
&= 4a^2x^3 + 2(1-2aw)x + (-2z) && (36)\\
&= 2\left(2a^2x^2 + (1-2aw)\right)x - 2z && (37)\\
\Leftrightarrow z &\overset{!}{=} (2a^2x^2 + (1-2aw))x && (38)\\
\Leftrightarrow 0 &\overset{!}{=} 2a^2x^3 + (1-2aw)x - z && (39)
\end{aligned}$$

The solution for this equation was computated with Wolfram|Alpha. I will only verify that the solution is correct. As there is only one solution in this case, we only have to check this one.

$$\begin{aligned}
t &:= \sqrt[3]{108a^4z + \sqrt{11\,664a^8z^2 + 864a^6(1-2aw)^3}} && (40)\\
x &= \frac{t}{6\sqrt[3]{2}a^2} - \frac{\sqrt[3]{2}(1-2aw)}{t} && (41)\\
\xRightarrow{x \text{ in line } 39} 0 &\overset{!}{=} 2a^2\left(\frac{t}{6\sqrt[3]{2}a^2} - \frac{\sqrt[3]{2}(1-2aw)}{t}\right)^3 + (1-2aw)\left(\frac{t}{6\sqrt[3]{2}a^2} - \frac{\sqrt[3]{2}(1-2aw)}{t}\right) - z && (42)\\
&= 2a^2\underbrace{\left(\frac{t}{6\sqrt[3]{2}a^2} - \frac{\sqrt[3]{2}(1-2aw)}{t}\right)^3}_{=:\alpha} + \frac{t\cdot(1-2aw)}{6\sqrt[3]{2}a^2} - \frac{\sqrt[3]{2}(1-2aw)^2}{t} - z && (43)
\end{aligned}$$

Now compute α. We know that $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$:

$$\begin{aligned}
\alpha &= \left(\frac{t}{6\sqrt[3]{2}a^2}\right)^3 - 3\left(\frac{t}{6\sqrt[3]{2}a^2}\right)^2\left(\frac{\sqrt[3]{2}(1-2aw)}{t}\right) + 3\left(\frac{t}{6\sqrt[3]{2}a^2}\right)\left(\frac{\sqrt[3]{2}(1-2aw)}{t}\right)^2 - \left(\frac{\sqrt[3]{2}(1-2aw)}{t}\right)^3 && (44)\\
&= \frac{t^3}{216\cdot 2\cdot a^6} - \frac{3t^2}{36\sqrt[3]{4}a^4}\cdot\frac{\sqrt[3]{2}(1-2aw)}{t} + \frac{3t}{6\sqrt[3]{2}a^2}\cdot\frac{\sqrt[3]{4}(1-2aw)^2}{t^2} - \frac{2(1-2aw)^3}{t^3} && (45)\\
&= \frac{t^3}{432a^6} - \frac{t(1-2aw)}{12\sqrt[3]{2}a^4} + \frac{\sqrt[3]{2}(1-2aw)^2}{2ta^2} - \frac{2(1-2aw)^3}{t^3} && (46)\\
&= \frac{t^3}{432a^6} + \frac{t^2(2aw-1) + 6\sqrt[3]{4}a^2(1-2aw)^2}{12\sqrt[3]{2}a^4} - \frac{2(1-2aw)^3}{t^3} && (47)\\
&= \frac{t^3 - 2(1-2aw)^3\cdot 432a^6}{432a^6t^3} + \frac{t^2(2aw-1) + 6\sqrt[3]{4}a^2(1-2aw)^2}{12\sqrt[3]{2}a^4} && (48)\\
&= \frac{\sqrt[3]{2}(t^3 - 2(1-2aw)^3\cdot 432a^6) + 36a^2t^3(t^2(2aw-1) + 6\sqrt[3]{4}a^2(1-2aw)^2)}{432\sqrt[3]{2}a^6t^3} && (49)\\
&= \frac{\sqrt[3]{2}t^3 - \sqrt[3]{2}(1-2aw)^3\cdot 864a^6 + 36a^2t^3(2awt^1 - t^2 + 6\sqrt[3]{4}a^2(1-2aw)^2)}{6\sqrt[3]{2}a^2t\cdot 2a^2\cdot 36a^2t^2} && (50)
\end{aligned}$$

So the solution is given by

$$x_S := -\frac{b}{2a} \quad \text{(the symmetry axis)}$$

$$\arg\min_{x\in\mathbb{R}} d_{P,f}(x) = \begin{cases} x_1 = +\sqrt{a(y_p + \frac{b^2}{4a} - c) - \frac{1}{2}} + x_S \text{ and} & \text{if } x_P = x_S \text{ and } y_p + \frac{b^2}{4a} - c > \frac{1}{2a} \\ x_2 = -\sqrt{a(y_p + \frac{b^2}{4a} - c) - \frac{1}{2}} + x_S & \\ x_1 = x_S & \text{if } x_P = x_S \text{ and } y_p + \frac{b^2}{4a} - c \leq \frac{1}{2a} \\ x_1 = todo & \text{if } x_P \neq x_S \end{cases}$$

5 Minimal distance to a cubic function

Let $f(x) = a \cdot x^3 + b \cdot x^2 + c \cdot x + d$ be a cubic function with $a \in \mathbb{R} \setminus \{ 0 \}$ and $b, c, d \in \mathbb{R}$ be a function.

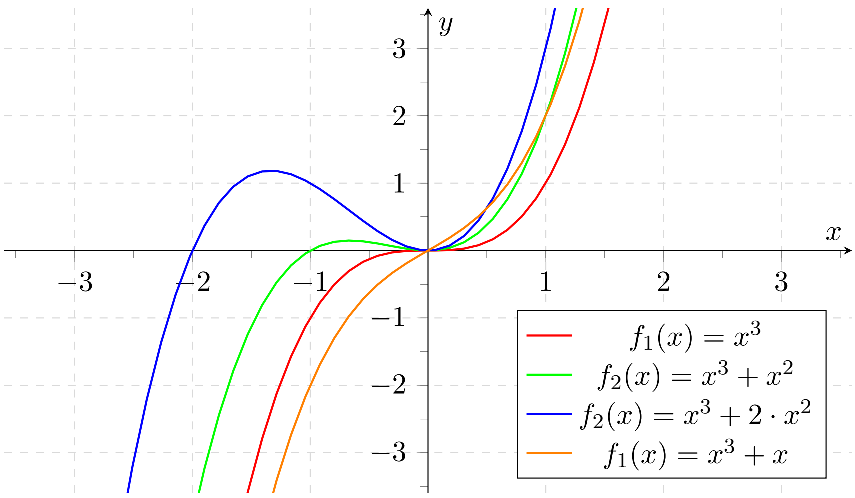

Figure 4: Cubic functions

5.1 Calculate points with minimal distance

When you want to calculate points with minimal distance, you can take the same approach as in Equation 13:

$$\begin{aligned}
0 &\stackrel{!}{=} -2x_p + 2x - 2y_p(f(x))' + (f(x)^2)' && (51)\\
&= 2f(x) \cdot f'(x) - 2y_p f'(x) + 2x - 2x_p && (52)\\
&= f(x) \cdot f'(x) - y_p f'(x) + x - x_p && (53)\\
&= \underbrace{f'(x) \cdot (f(x) - y_p)}_{\text{Polynomial of degree 5}} + x - x_p && (54)
\end{aligned}$$

General algebraic equations of degree 5 don't have a solution formula.[1] Although here seems to be more structure, the resulting algebraic equation can be almost any polynomial of degree 5:[2]

$$\begin{aligned}
0 &\stackrel{!}{=} f'(x) \cdot (f(x) - y_p) + (x - x_p) && (55)\\
&= \underbrace{3a^2}_{=\tilde{a}} x^5 + \underbrace{5ab}_{\tilde{b}} x^4 + \underbrace{2(2ac + b^2)}_{=:\tilde{c}} x^3 \quad + \underbrace{3(ad + bc - ay_p)}_{\tilde{d}} x^2 && (56)\\
&\qquad + \underbrace{(2bd + c^2 + 1 - 2by_p)}_{=:\tilde{e}} x + \underbrace{cd - cy_p - x_p}_{=:\tilde{f}} && (57)\\
0 &\stackrel{!}{=} \tilde{a}x^5 + \tilde{b}x^4 + \tilde{c}x^3 + \tilde{d}x^2 + \tilde{e}x + \tilde{f} && (58)
\end{aligned}$$

1. With a, we can get any value of $\tilde{a} \in \mathbb{R} \setminus \{ 0 \}$.

[1] TODO: Quelle

[2] Thanks to Peter Košinár on math.stackexchange.com for this one

2. With b, we can get any value of $\tilde{b} \in \mathbb{R} \setminus \{ 0 \}$.
3. With c, we can get any value of $\tilde{c} \in \mathbb{R}$.
4. With d, we can get any value of $\tilde{d} \in \mathbb{R}$.
5. With y_p, we can get any value of $\tilde{e} \in \mathbb{R}$.
6. With x_p, we can get any value of $\tilde{f} \in \mathbb{R}$.

The first restriction only guaratees that we have a polynomial of degree 5. The second one is necessary, to get a high range of $\tilde{e}$.

This means, that there is no solution formula for the problem of finding the closest points on a cubic function to a given point.

5.2 Another approach

Just like we moved the function f and the point to get in a nicer situation, we can apply this approach for cubic functions.

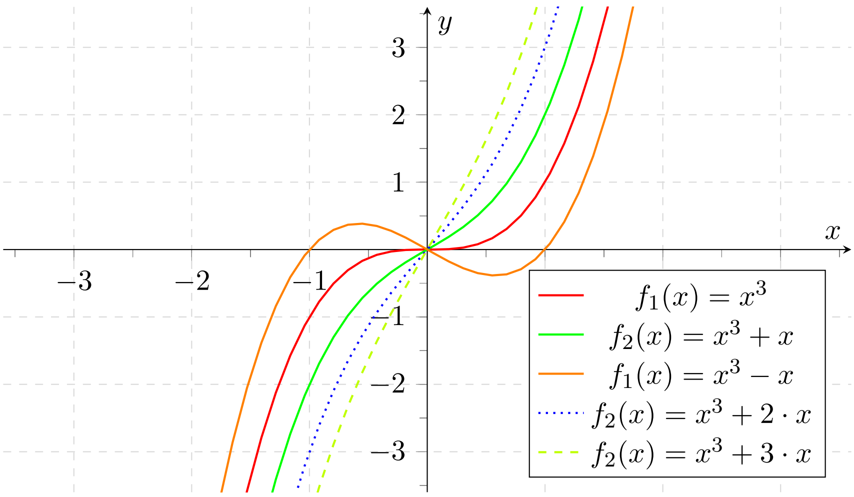

Figure 5: Cubic functions with $b = d = 0$

First, we move f_0 by $\frac{b}{3a}$ to the right, so

$$f_1(x) = ax^3 + \frac{b^2(c-1)}{3a}x + \frac{2b^3}{27a^2} - \frac{bc}{3a} + d \quad \text{and} \quad P_1 = (x_P + \frac{b}{3a}, y_P)$$

because

$$
\begin{aligned}
f_1(x) &= a\left(x - \frac{b}{3a}\right)^3 + b\left(x - \frac{b}{3a}\right)^2 + c\left(x - \frac{b}{3a}\right) + d && (59)\\
&= a\left(x^3 - 3\frac{b}{3a}x^2 + 3(\frac{b}{3a})^2x - \frac{b^3}{27a^3}\right) + b\left(x^2 - \frac{2b}{3a}x + \frac{b^2}{9a^2}\right) + cx - \frac{bc}{3a} + d && (60)\\
&= ax^3 - bx^2 + \frac{b^2}{3a}x - \frac{b^3}{27a^2} && (61)\\
&\quad + bx^2 - \frac{2b^2}{3a}x + \frac{b^3}{9a^2} && (62)\\
&\qquad + cx - \frac{bc}{3a} + d && (63)\\
&= ax^3 + \frac{b^2}{3a}(1 - 2 + c)\,x + \frac{b^3}{9a^2}\left(1 - \frac{1}{3}\right) - \frac{bc}{3a} + d && (64)
\end{aligned}
$$

Which way to move might be clever?

5.3 Number of points with minimal distance

As there is an algebraic equation of degree 5, there cannot be more than 5 solutions.

Can there be 3, 4 or even 5 solutions? Examples!
After looking at function graphs of cubic functions, I'm pretty sure that there cannot be 4 or 5 solutions, no matter how you chose the cubic function f and P.
I'm also pretty sure that there is no polynomial (no matter what degree) that has more than 3 solutions.

If there is no closed form solution, I want to describe a numerical solution. I guess Newtons method might be good.