Theorem 1 *Let $Y \sim \mathcal{N}(\mu, \sigma^2)$ and $X \sim e^Y$. Then X has the density*

$$f_X(x) = \begin{cases} \frac{1}{x\sigma\sqrt{2\pi}} \exp -\frac{(\log x - \mu)^2}{2\sigma^2} & \textit{if } x > 0 \\ 0 & \textit{otherwise} \end{cases}$$

Proof:

$$\begin{aligned} \mathbb{P}(X \leq t) &= \mathbb{P}(e^Y \leq t) && (1) \\ &= \begin{cases} \mathbb{P}(Y \leq \log(t)) & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases} && (2) \end{aligned}$$

Obviously, the density $f_X(x) = 0$ for $x \leq 0$. Now continue with $t > 0$:

$$\begin{aligned} \mathbb{P}(X \leq t) &= \mathbb{P}(Y \leq \log(t)) && (3) \\ &= \Phi_{\mu,\sigma^2}(\log(t)) && (4) \\ &= \Phi_{0,1}\left(\frac{\log(t) - \mu}{\sigma}\right) && (5) \\ f_X(x) &= \frac{\partial}{\partial x}\Phi_{0,1}\left(\frac{\log(x) - \mu}{\sigma}\right) && (6) \\ &= \left(\frac{\partial}{\partial x}\left(\frac{\log(x) - \mu}{\sigma}\right)\right) \cdot \varphi_{0,1}\left(\frac{\log(x) - \mu}{\sigma}\right) && (7) \\ &= \left(\frac{\sigma \cdot \frac{1}{x}}{\sigma^2}\right) \cdot \varphi_{0,1}\left(\frac{\log(x) - \mu}{\sigma}\right) && (8) \\ &= \frac{1}{x\sigma} \cdot \varphi_{0,1}\left(\frac{\log(x) - \mu}{\sigma}\right) && (9) \\ &= \frac{1}{x\sigma} \cdot \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} \cdot \left(\frac{\log(x) - \mu}{\sigma}\right)^2\right) && (10) \end{aligned}$$

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